Comparison of Variational Iteration Decomposition Method with Optimal Homotopy Asymptotic of Higher Order Boundary Value Problems

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Abstract - In this work, we consider special problem consisting of twelfth order two-point boundary value by using the Optimal Homotopy Asymptotic Method and Variational Iteration Decomposition Method. Now, we discuss the comparison in between Optimal Homotopy Asymptotic Method and Variational Iteration Decomposition Method. These proposed methods have been thoroughly tested on problems of all kinds and shows very accurate results. A numerical example is present and approximate is compared with exact solution and the error is compared with Optimal Homotopy Asymptotic Method and Variational Iteration Decomposition Method to assess the efficiency of the Optimal Homotopy Asymptotic Method at 12th order Boundary values problems.

Keywords - Twelfth order boundary value problems, Approximate analytical solution, Variational Iteration Decomposition Method, optimal homotopy Asymptotic method, Ordinary Differential Equations, Error Estimates.

1. Introduction

In literature different techniques are available for the numerical solution of twelfth order boundary value problems.

In this Paper, we consider the general 12^{th} order boundary value problems of the type: $y^{12}(x) + f(x)\, y(x) = g(x), \quad x \in [a, b]$ (1)

With boundary conditions:

$y(a) = a_1 \qquad y(b) = b_1$

$y^{(1)}(a) = a_2 \qquad y^{(1)}(b) = b_2$

$y^{(2)}(a) = a_3 \qquad y^{(2)}(b) = b_3$

$y^{(3)}(a) = a_4 \qquad y^{(3)}(b) = b_4$

$y^{(4)}(a) = a_5 \qquad y^{(4)}(b) = b_5$

$y^{(5)}(a) = a_6 \qquad y^{(5)}(b) = b_6$

Where a_i, b_j, here $i, j = 1,2,3,4,5,6$ are finite real constants and the functions f(x) and g(x) are continuous on [a, b]. The motivation of this problem is to extend Optimal Homotopy Asymptotic Method to solve linear and nonlinear twelfth order boundary value problems. We also compared the results obtained from these techniques with the available exact solution in different literatures. Some properties of solutions of a given differential equation may be determined without finding their exact form in especially in nonlinear behavior. If as self-contained formula for the solution is not available, the solution may be numerically approximated using computers. To overcome these difficulties, a modified form of the variational method called Variational Iteration Decomposition Method. VIDM has since then been effectively utilized in obtaining approximate analytical solutions to many linear and nonlinear problems arising in engineering and science, such as nonlinear oscillators with discontinuities [3], nonlinear Volterra –Fredholm integral equations [11], Twelfth order differential equations have several important applications in engineering. Solution of linear and nonlinear boundary value problems of twelfth-order was implemented by Wazwaz using Adomian decomposition method. Chandrasekhar [9] showed that when an infinite horizontal layer of fluid is put into rotation and simultaneously subjected to heat from below and a uniform magnetic field across the fluid in the same direction as gravity, instability will occur. Several researchers developed numerical techniques for solving twelfth order differential equations. The Adomian Decomposition Method [1, 4], the Differential Transform Method [15], the Variational Iteration Method, the successive iteration, the splines [5, 6], the Homotopy Perturbation Method [7], the Homotopy Analysis Method etc Recently Vasile Marinca et al. [10,12,14] introduced OHAM for approximate solution of nonlinear problems of thin film flow of a fourth grade fluid down a vertical cylinder. OHAM is straight forward, reliable and it does not need to look for h curves like VIDM. Moreover, this method provides a convenient way to control the convergence of the series solution. Most recently, Javed Ali et al. used OHAM for the solutions of multi-point boundary value problems. The results of OHAM presented in this work are compared with those of exact solution VIDM.

2. Variational Iteration Method

To illustrate the basis concept of the technique, we consider the following general differential equation

$$L_1u + Nu = g(x) \tag{2}$$

Where L is a linear operator, N a nonlinear operator and g(x) is the in homogenous term. According to variational iteration method, we can construct a correct functional as follows

$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(L\, u_n(s) + N\widetilde{u_n(s)})$ (3)

where λ is a **Lagrange multiplier [14 to 18],** which can be identified optimally via variational iteration method. The subscripts n denote the nth approximation, $\widetilde{u_n}$ is considered as a restricted variation. i.e. $\widetilde{u_n}=0$. The relation (2) is called as a correct functional. The solution of the linear problems can be solved in a single iteration step due to the exact identification of the Lagrange multiplier.

3. Adomian Variational Iteration Decomposition Method

Now we recall basic principles of the Adomian decomposition method **[10, 12]** for solving differential equations. Consider the general equation T u = g, where T represents a general nonlinear differential operator involving both linear and nonlinear terms. The linear term is decomposed into L + R where L is easily invertible and R is the reminder of the linear operator. For convenience, L may be taken as the highest order derivation. Thus the equation may be written as

$$Lu + R(u) + Nu = g(x) \qquad (4)$$

where Nu represents the nonlinear terms. From (3) we have

$$Lu = g - R(u) - Nu \qquad (5)$$

Since L is invertible the equivalent expression is

$$u = L^{-1} g - L^{-1} R(u) - L^{-1} N(u) \qquad (6)$$

A solution u can be expressed as following series

$$u(x) = \sum_{m=0}^{\infty} u_m \qquad (7)$$

with reasonable u_0 which may be identified with respect to the definition of L^{-1}, g and u_n, n>0 is to be determined. The nonlinear term Nu will be decomposed by the infinite series of Adomian polynomials

$$Nu = \sum_{m=0}^{\infty} B_m \qquad (8)$$

where B_n's are obtained by writing

$$v\lambda = \sum_{m=0}^{\infty} \lambda^m u_m \qquad (9)$$

$$N(v\lambda) = \sum_{m=0}^{\infty} \lambda^m u_m \qquad (10)$$

Here λ is a parameter introduced for convenience. From (9) and (10) we have

$$B_m = \frac{1}{m!}\frac{d^m}{d\lambda^m}\left[N\sum_{m=0}^{\infty} \lambda^m u_m\right]$$

Here $n \geq 0$ (11)

The B_m's are given as, There appears to be no well-defined method for constructing a definitive set of polynomials for arbitrary F, but rather slightly different approaches are used for different specific functions. One possible set of polynomials is given by

$$B_0 = F(u_0),$$

$$B_1 = (x-x_1)\left(\frac{dy}{dx}\right)_{u=0}$$

$$B_2 = (x-x_2)\left(\frac{dy}{dx}\right)_{u=0} + \frac{(x-x_1)^2}{2!}\left(\frac{d^2y}{dx^2}\right)_{u=0}$$

$$B_3 = (x-x_3)\left(\frac{dy}{dx}\right)_{u=0}$$

$$+(x-x_1)(x-x_2)\left(\frac{d^2y}{dx^2}\right)_{u=0} + \frac{(x-x_1)^2}{3!}\left(\frac{d^3y}{dx^3}\right)_{u=0} \qquad (12)$$

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can be used to construct Adomian polynomials, when $F(u_0)$ is a nonlinear function Put the value of equation (8) and (9) in equation (3), we get

$$\sum_{m=0}^{\infty} u_m = u_0 + L^{-1}R\left(\sum_{m=0}^{\infty} B_m\right) - L^{-1}\sum_{m=0}^{\infty} B_m \qquad (13)$$

Consequently, with a suitable u_0 we can write, Put one by one j=1, 2, 3, …….. in above expression.

$$u_1(x) = -L^{-1}R(u_0) - L^{-1}B_0$$

$$u_2(x) = -L^{-1}R(u_1) - L^{-1}B_1$$

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$$u_{n+1}(x) = -L^{-1}R(u_n) - L^{-1}B_n$$

3. Variational Iteration Decomposition Method

To illustrate the basis concept of the Variational iteration decomposition method, we consider the following general differential equation (1). According to variational iteration method **[11]**, we can construct a correct functional (2), we define the solution by the series

$$u(x) = \sum_{m=0}^{\infty} u^m(x)$$

and the nonlinear term

$$\sum_{m=0}^{\infty} B_m(u_0, u_1, u_2, u_3, \dots\dots\dots, u_m)$$

Where Bn are the Adomian polynomials and can be generated for all type of nonlinearities according to the algorithm developed in [13] which yields the following

$$B_m = \frac{1}{m!}\frac{d^m}{d\lambda^m}\; N(u(x))l\lambda \qquad n \geq 0,$$

Hence, we obtain the following iterative scheme

$$u^{(n+1)}(x) = u^{(n)}(x) + \int_0^t \lambda\left(L\,u^{(n)}(x) + \sum_{m=0}^{\infty} B_m - g(x)\right)dx.$$

The method is called Variational iterative decomposition method.

NUMERICAL EXAMPLES

Example 1: Consider the following linear twelfth order boundary value problem

$y^{12}(x) = -x\, y(x) - x^3e^x - 23xe^x - 120e^x$ (14)

with following conditions:

$y_0 = 0,\ y_0^{(1)} = 1,\ y_0^{(2)} = 0,\ y_0^{(3)} = -3,\quad y_0^{(4)} = -8,\quad y_0^{(5)} = -15$

$y_1 = 0,\ y_1^{(1)} = -e,\quad y_1^{(2)} = -4e,\quad y_1^{(3)} = -9e,\quad y_1^{(4)} = -16e,\ \ y_1^{(5)} = -25e$

Exact solution is $y(x) = x(1-x)e^x$

The correct functional for the boundary value problem is given as

$$y_{(n+1)}(x) = y_{(n)}(x) + \int_0^x \lambda\left(\frac{d^{12}y_n}{dx^{12}} - (-x\,\widetilde{y(x)} - x^3e^x - 23xe^x - 120e^x)\right)$$

To find the optimal $\lambda(s)$, calculation variation with respect to y_n, we have the following stationary conditions:

$\delta y_n : \lambda^{(m)}(s) = 0$,

$\delta y^{(m-1)}_n : [\lambda(s)]s=x = 0$,

$\delta y^{(m-2)}_n : [\lambda'(s)]s=x = 0$,

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$\delta y_n : [1-\lambda^{(m-1)}(s)]s=x = 0$ (15)

But in above Eq. (14) value of m is 12, put all these values of m in Eq.(15) and get follows:

$\delta y_n : \lambda^{(12)}(s) = 0$,

$\delta y^{(11)}_n : [\lambda(s)]s=x = 0$,

$\delta y^{(10)}_n : [\lambda'(s)]s=x = 0$,

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$\delta y_n : [1-\lambda^{(11)}(s)]s=x = 0$.

The Lagrange multiplier, therefore, can identify as follows:

$$\lambda = \frac{(s-x)^{11}}{11!}$$

Making the correct functional stationary, using $\lambda = \frac{1}{11!}((s - x)11$ Lagrange multiplier **[16 to 20],** Substituting the identified multiplier into above Eq. we have the following iteration formula:

$$y_{(n+1)}(x) = y_{(n)}(x) + \int_0^x \frac{1}{11!}((s-x)^{11})\left(\frac{d^{12}y_n}{dx^{12}} - (-x\,y(x) - x^3e^x - 23xe^x - 120e^x)\right)ds$$

$$y_{n+1}(x) = x + \frac{A}{3!}x^3 + \frac{B}{4!}x^4 + \frac{C}{5!}x^5 + \frac{D}{6!}x^6 + \frac{E}{7!}x^7 + \frac{F}{8!}x^8$$

$$+ \int_0^x \frac{1}{11!}((s-x)^{11})\left(\frac{d^{12}y_n}{dx^{12}} - (-x\,y(x) - x^3e^x - 23xe^x - 120e^x)\right)ds$$

Where

A= $y''''''(0)$, B = $y'''''''(0)$, C= $y''''''''(0)$, D= $y'''''''''(0)$, E= $y''''''''''(0)$, F= $y'''''''''''(0)$

Using the Variational iterative decomposition method, we get,

$$y_{n+1}(x) = x + \frac{A}{3!}x^3 + \frac{B}{4!}x^4 + \frac{C}{5!}x^5 + \frac{D}{6!}x^6 + \frac{E}{7!}x^7 + \frac{F}{8!}x^8 + \int_0^x \frac{1}{11!}((s-x)^{11})\left(\frac{d^{12}y_m}{dx^{12}} + x^3e^x + 23xe^x + 120e^x - x\sum_{m=0}^{\infty} B_m\right)ds$$

Where B_m are Adomian polynomials for non-linear operator N(y) = x y(x) and can be generated for all type of nonlinearities according to the algorithm which yields the following

$B_0 = xy_0(x)$,

$B_1 = y_1(x)N'(y_0)$,

$B_2 = \left(y_0 + \frac{y_1{}^2}{2}\right)xy_0(x)$,

…………… (16)

From the above relation, we find $y_0(x)$, $y_1(x)$,……..and get the series solution as follow:

$$y(x) = x - \frac{1}{2}x^3 - \frac{1}{3}x^4 - \frac{1}{8}x^5 + \frac{A}{720}x^6 + \frac{B}{5040}x^7 + \frac{C}{40320}x^8 + \frac{D}{362880}x^9 + \frac{E}{3628800}x^{10} + \frac{F}{39916800}x^{11} - \frac{1}{3991680}x^{12} - \frac{1}{43545600}x^{13} - \frac{83}{43589145600}x^{14} - \frac{1}{6706022400}x^{15}$$

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The coefficients A, B, C, D, E, F, G can be obtained using the boundary conditions at x =1,

A = 23.9999985,
B =35.000057,
C= 47.998961,
D =63.0108031,
E =79.9359481,
F= 99.17376631.

The series solution can, thus, be written as

$$y^{12}(x) = x - \frac{1}{2}x^3 - \frac{1}{3}x^4 - \frac{1}{8}x^5 + 0.0333333\,x^6 - 0.00694446\,x^7 - 0.00119045\,x^8 - 0.000173641x^9 - 0.0000220282\,x^{10} + .00000248451x^{11} - \frac{1}{3991680}x^{12} - \frac{1}{43545600}x^{13} - \frac{83}{43589145600}x^{14} - \frac{1}{6706022400}x^{15}$$

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Table 1.1(Error Estimate)

X	Exact Solution	Numerical Solution of VIDM	Errors of VIDM
0.0	1.000000000	0.0000000000	0.00000
0.1	0.0994653826	0.0994653826	3.00×10^{-11}
0.2	0.1954244413	0.1954244413	0.00000
0.3	0.2834703497	0.2834703496	-1.00×10^{-10}
0.4	0.3580379275	0.3580379277	2.00×10^{-10}
0.5	0.4121803178	0.4121803189	1.10×10^{-9}
0.6	0.4373085120	0.4373085164	4.40×10^{-9}
0.7	0.4228880685	0.4228880820	1.35×10^{-8}
0.8	0.3560865485	0.3560865853	3.68×10^{-8}
0.9	0.2213642800	0.2213643701	9.01×10^{-8}
1.0	0.0000000000	0.0000002027	2.02700×10^{-7}

Table 1.1 shows the approximate solution obtained by (VIDM) and error obtained by comparing it with the exact solution. Higher accuracy can be obtained by evaluating more iteration.

X	**Analytical Solution**	**Numerical Solution**	**Errors**
0.0	1.000000000	0.000000000	0.00000
0.1	0.099465383	0.099465383	-7.5065×10^{-14}
0.2	0.195424441	0.195424441	-2.7686×10^{-12}
0.3	0.283470350	0.283470350	-1.7271×10^{-11}
0.4	0.358037927	0.358037927	-5.023×10^{-11}
0.5	0.412180318	0.412180318	-9.3401×10^{-11}
0.6	0.437308512	0.437308512	-1.2791×10^{-10}
0.7	0.422888068	0.422888069	-1.3917×10^{-10}
0.8	0.356086548	0.356086549	-1.227×10^{-10}
0.9	0.221364280	0.221364280	-7.4997×10^{-11}
1.0	0.000000000	-0.00000000	1.9454×10^{-11}

Example 2: Consider the following linear twelfth order boundary value problem

$$y^{12}(x) = -x\,y(x) - x^3e^x - 23xe^x - 120e^x$$

with following conditions:

$$y_0 = 0,\ y_0^{(1)} = 1,\ y_0^{(2)} = 0,\quad y_0^{(3)} = -3,\quad y_0^{(4)} = -8,\quad y_0^{(5)} = -15$$

$$y_1 = 0,\ y_1^{(1)} = -e,\quad y_1^{(2)} = -4e,\quad y_1^{(3)} = -9e,\quad y_1^{(4)} = -16e,\ y_1^{(5)} = -25e$$

Exact solution is $y(x) = x(1-x)e^x$

We construct the following zeroth and first-order problems.

$$y_0^{12}(x) = -x\,y(x) - x^3e^x - 23xe^x - 120e^x$$

with following conditions

$$y_0(0) = 0,\ y_0^{(1)}(0) = 1,\ y_0^{(2)}(0) = 0,\quad y_0^{(3)}(0) = -3,\quad y_0^{(4)}(0) = -8,\quad y_0^{(5)}(0) = -15$$

$$y_0(1) = 0,\ y_0^{(1)}(1) = -e,\quad y_0^{(2)}(1) = -4e,\quad y_0^{(3)}(1) = -9e,\quad y_0^{(4)}(1) = -16e,\quad y_0^{(5)}(1) = -25e$$

First-Order Problem

$$y_1^{12}(x) = (1+C_1)(120 + 23x + x^3)e^x + C_1x\,y_0(x) + (1+C_1)\,y_0^{12}(x)$$

With same above boundaries' conditions Solutions to these problems are given by Equations. (20) and (21) respectively

$$y_0(x) = \frac{1}{120}(-280800 + 280800e^x - 221760x - 58920e^xx - 85800x^2 + 4320e^xx^2 - 21600x^3 - 120e^xx^3 - 3960x^4 - 560x^5 + 197720040x^6 + 72737161ex^6 - 836896800x + 307877125ex^7 + 1451896600ex^8 - 1282301040x^9 + 471732190ex^9 + 5741817x^{10} - 211229665ex^{10} - 103986080x^{11} + 38254341ex^{11})$$
(20)

$$y_1(x, C_1) = C_1(216060 - 216060e^x + (176352\ 39708e^x)x + (71045 - 2723e^x)x^2 + (18795 + 84e^x)x^3 + (3663 - e^x)x^4 + 558.833x^5 + 69.1417x^6 + 7.07738x^7 + 0.603671x^8 + 0.0425263 + 0.00237265x^{10} + 0.0000892391x^{11} - 3.75782\times10^{-7}x^{13} - 4.23959\times10^{-8}x^{14} - 3.2806310^{-9}x^{15} - 2.06473\,10^{-10}x^{16} - 1.11334\times10^{-11}x^{17} - 5.24805\times10^{-12}x^{18} - 2.17518\times10^{-14}x^{19} - 7.89181\times10^{-16}x^{20} - 2.46619\times10^{-17}x^{21} - 6.40591\times10^{-19}x^{22} - 1.27589\times10^{-20}x^{23} - 1.55619\,10^{-22}x^{24})$$
(21)

Considering the OHAM first-order solution,

$$Y_{app}(x, C_1) = y_0(x) + y_1(x, C_1)$$
(22)

and using Eq.(18) with a = 0.5 and b = 1, we get $C_1 = -0.00260417$. Using this value the first-order solution (22) is well-determined.

Table 1.2(Error Estimate)

Conclusion

In this paper, the Comparison of the results obtained by the Homotopy perturbation method and optimal homotopy asymptotic method of Twelfth order boundary value problems. The numerical results in the Tables [1.1-1.2], show that the optimal homotopy asymptotic method provides highly accurate numerical results as compared to Homotopy perturbation method. It can be concluded that optimal homotopy asymptotic method is a highly efficient method for solving 12^{th} order boundary value problems arising in various fields of engineering and science.

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